Further Applications of P-Adaptive Boundary Elements

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ABSTRACT

In previous BEM Conferences, the concepts, developments and organisation of the p-adaptive philosophy have been presented by the authors, as well as some interesting features of the hierarchisation of the solution, accuracy estimates and numerical computations optimization.

This current paper is devoted to presenting some new developments and applications in linear elastostatics, with emphasis on: a) Efficient computation of influence coefficients, b) Efficient evaluation of the residuals by taking advantage of the hierarchy of the interpolation functions and c) New results regarding estimators and convergence ratios.

In addition, several practical examples will be shown and discussed in order to point out the advantages of the method.

INTRODUCTION

In previous papers |1,2,8,9| both the mathematical formulae and the refinement criteria needed for the development and implementation of the p-adaptive BEM version were presented by the authors.

The advantages of the boundary methods and the spectacular outcomes obtained in the applications of the self-adaptive FEM techniques |3,12,13,17,26| merged together have shown to be very attractive in the analysis of potential and elastostatics problems. The reader interested in a survey of p-adaptive FEM is refered to |5|.

Rank |19| and Rencis et. al. |22| have succesfully applied the BEM h-version to 2-D elasticity. Also, Parreira |15| and Rank |20| have presented a p-adaptive BEM version in 2-D elastostatics. These authors, however, make use of straight boundary elements and analytical integration of the kernel functions, which diminish the compliance nature of the method. The analytical evaluation of the singular integrals in the Cauchy Principal Value (CPV) sense also causes a loss of generality in the method. This is one of the reasons why the available existing BEM codes make use of numerical cuadratures for the integrals which normally arise in boundary integral methods.

Thus, in this work we have decided to use curved variable-number-nodes boundary elements and fully numerical integration for the evaluation of both the singular kernels and the mathematical criteria which govern the self-adaptive refinement process.

SOME REMARKS REGARDING THE P-ADAPTIVE BEM VERSION

As it is well known, the advantages of self-adaptive techniques basically lie in: a) the possibility of performing the boundary discretisation with macro-elements compatible with both the geometry and boundary conditions, thus reducing human effort in data preparation and b) the estimation of the error produced by the numerical solution.

The first step in the p-adaptive version is the choice of a suitable rough initial mesh, which does not violate the requirements imposed by the boundary geometry and the boundary conditions. Further polynomial refinements are to be done based upon this initial mesh. The rationale behind all of this is that, with the exception of corners and sudden changes in boundary conditions, the data and unknown fields on the remaining elements must behave as smooth functions. As a consequence, a hierarchical approach in the sense of a Fourier series development, shall represent the behaviour of such functions with only a limited number of extra degrees of freedom.

Next, the hierarchy of the interpolation functions to be used in the successive approximation process is defined. In this research we have decided to employ -

the Legendre family |12| due to the fact that these - functions produce the best matrix conditioning in the method.

On the other hand, the introduction of new interpolation functions presupposes the previous choice of new points in order to place the fundamental solution. This set of source points should be selected in such a way as to satisfy, at least, the following two recommendations |8,9|: a) they should be as far apart as - possible from previous source points in order to avoid ill-conditioning of the influence matrix and b) the - new equation must reinforce the corresponding dominant element within the influence matrix.

The criteria to evaluate the error done in the - approximate solution are, undoubtedly, the corner stone in the self-adaptive schemes of any method. Thus, it becomes necessary to arrange local error indicators and global estimators in order to be able to decide "where" and "in what way" a further refinement on the initial mesh is needed. A brief summary of such mathematical criteria is given in the following section.

LOCAL ERROR INDICATORS AND GLOBAL ESTIMATORS

The final formulae developed by the authors for both the evaluation of the numerical error and how to guide the p-adaptive process is included herein. The reader interested in further details may see for instance, references |1,2,9,11|.

The local error indicator which governs the refinement process may be written as

$$||e||^2_{kj} = \frac{\left[\int_{S_k} H_j^{n+1}(Q)\, r_j(Q)\, ds(Q)\right]^2}{\int_{S_k} H_j^{n+1}(Q)\, L_j(P,Q)\, H_j^{n+1}(Q)\, ds(Q)} \tag{1}$$

- k = boundary element which is being refined in the present step.
- j = unknown variable which is being refined on the element k.
- r_j = residual function obtained on element k.
- H_j^{n+1} = new interpolation function to be included in the present step.
- L_j = vectorial integral operator based on Somigliana's identity.

Expression (1) allows us to determine what elements and which variables need to be refined, either in an automatic or an interactive way.

The residual function $\underset{\sim}{r}$ could be written as

$$\underset{\sim}{r}(P) = \underset{\sim}{C}(P) \left[\hat{\underset{\sim}{u}}(P) - \underset{\sim}{u}^{COMP}(P)\right] \tag{2}$$

where $\underset{\sim}{C}$ is a matrix related to local geometric peculiarities of the boundary around the source point |6|. $\hat{\underset{\sim}{u}}$ is the numerical solution obtained by interpolation of the actual variables inside the elements which are to be refined. Also, $\underset{\sim}{u}^{COMP}$ are the computed values obtained when the fundamental solution is placed in the same points described above.

Finally, the global error estimator is calculated by means of the Ho Hilbert norm (see, e.g., Reddy |21|) which has the following form

$$||E||_o = \left[\int_S \sum_{i=1}^{N_d} r_i^{\,2}(P)\, ds(P)\right]^{1/2} \tag{3}$$

where the integral in (3) is now extended over the whole boundary and Nd is the dimensionality of the problem considered (Nd=1,2,3 in potential, 2-D elastostatics and 3-D elastostatics problems, respectively).

The particular behaviour of the residual function $\underset{\sim}{r}$, which vanishes in all previous source points, suggests at once the possibility of its numerical evaluation through the Lobatto's numerical cuadrature (see, e.g., Stroud and Secrest |24|). This numerical cuadrature, as it is well known, evaluates a regular function provided that its values are known at the outermost points of the integration gap. Thus, by remembering that $r_i(\xi=-1)=0$ and $r_i(\xi=1)=0$ it is possible to write

$$\int_{-1}^{1} \left(\sum_{i=1}^{N_d} r_i^{\,2}\right) J_{(\xi)}\, d\xi \simeq \sum_{j=2}^{N-1} \left(\sum_{i=1}^{N_d} r_{i(\xi_j)}^{\,2}\right) J_{(\xi_j)}\, \omega_j \tag{4}$$

where the ξ_j and the ω_j are the abscissas and weighting factors of Lobatto's cuadrature. Our experience has shown that a numerical value of N=5 provides - enough accuracy for practical purposes in the p-adaptive BEM approach, thus requiring only three residual evaluations inside the integration gap.

The numerical evaluation of the local indicators as well as the fundamental solution itself involves the integration of nearly singular kernels in the sense of CPV. In the p-adaptive BEM version, this point is critical for the accuracy of the method, in part due to the large size of the boundary elements chosen for the discretisation.

The integration of nearly singular kernels (i.e., those in which the source point is very close to the integration gap), produces unreliable results if special precautions are not taken. Telles |25| has developed a non-linear coordinate transformation which gathers the sampling points nearby the source point, thus giving great accuracy. Therefore, Telles' scheme has been incorporated into the integration routines developed in this research.

In the case of singular kernels (i.e., those in which the source point belongs to the integration gap), a bi-cubic coordinate transformation has been developed |10,11|. Such a transformation is closely related to the theoretical definition of CPV integrals and it notably increases the degree of accuracy with a reasonable number of integration points, even in the presence of boundary elements of very different sizes meeting a source point. Moreover, this approach does not practically require changes in the available integration routines and makes use of standard gaussian cuadratures.

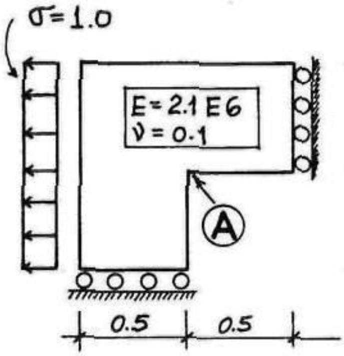
CASE STUDIES: ANALYSIS AND DISCUSSION

Two numerical examples which illustrate the performance of the p-adaptive BEM version in elastostatics are presented and discussed here. These examples include singularities due to geometry and sudden changes in boundary conditions.

L-shaped domain

Figure 1.a shows the geometry and boundary conditions of an L-shaped plate subjected to boundary tractions, whereas figure 1.b shows the initial mesh employed in the p-refinement, defined by six macroelements whose numbers are circled. The significance of this case lies in the presence of a weak geometric singularity (corner A) where the stress fields have a tendency towards infinite values |12|.

This problem was analysed with both a 120 boundary element mesh (20 elements/side) and the p-refinement



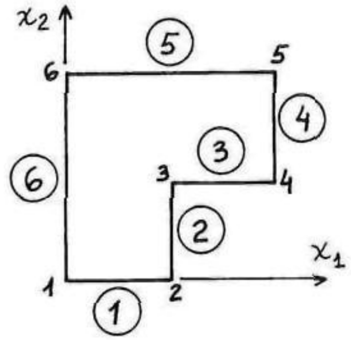


Figure 1.a
Geometry and boundary conditions.

Figure 1.b
p-adaptive discretisation

proposed herein. Figure 2 collects the numerical results for the stress σ_1 calculated along the vertical line $x_1=0.5$.

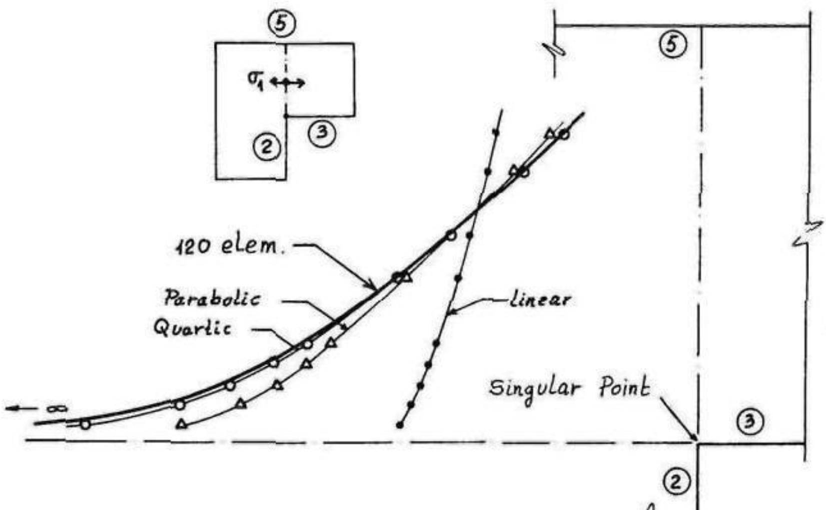


Figure 2
Stresses σ_1 on $x_1=0.5$. L-shaped plate

The thick line representsthe 120-element solution whereas the lines with symbols show the stresses obtained in some p-refinement steps. As can be seen, the p-adaptive quartic step gives rather satisfactory results when compared with the 120-element solution.

The global convergence rates of the plate are - shown in figure 3. An h-refinement, which gives a convergence rate of 0.70, was performed over the base - mesh in order to be compared with the p-refinement - ones. The p-adaptive convergence rate (line with circles) was 2.0 (i.e., 2.87 times faster than the h-refinement one). The p-complete approach (introduction of all interpolation functions in each step) gave a convergence rate of 1.60 times faster than h-refinement one, but it was slower than the p-adaptive one. This fact shows that an effective selective criteria must be used in order to decide which new interpolation - functions must be included to improve the convergence rate.

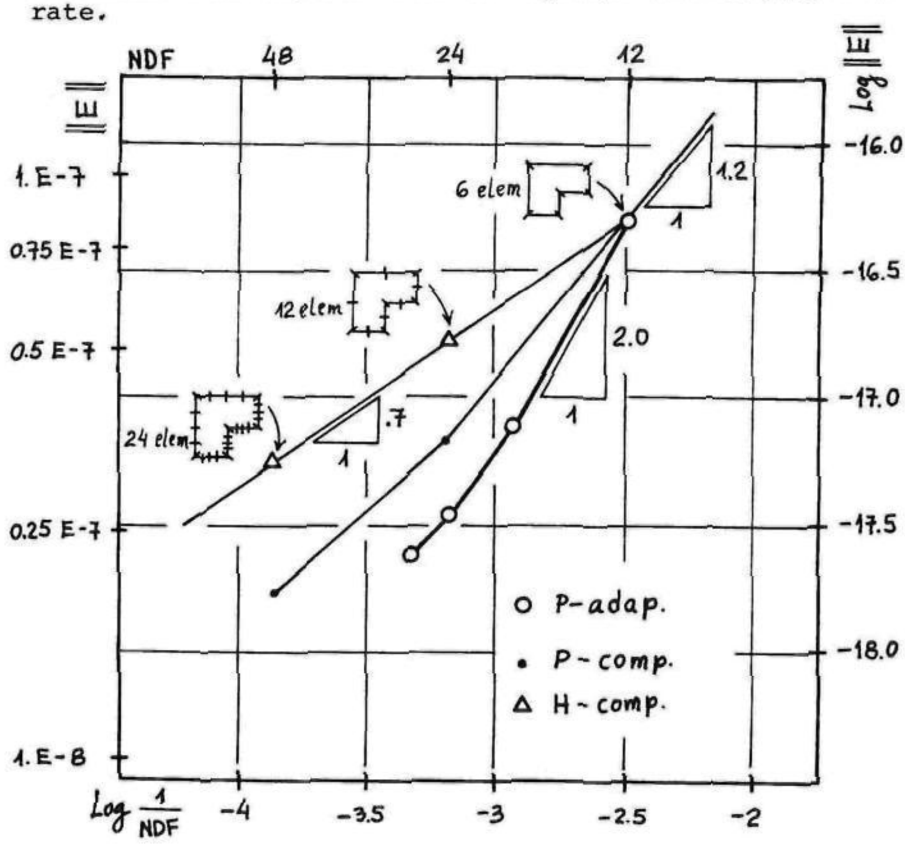


Figure 3
Comparison between global convergence rates.
L-shaped plate.

Also note that the geometric singularities (those which do not involve sudden changes in boundary conditions) can be effective treated with the p-adaptive - approach. Observe that the singular point A is bounded by only two macro-elements, it not being necessary to define additional small elements in order to catch the stress concentration.

Cracked plate.

This example is typical within the context of fracture mechanics |7|: the evaluation of the stress fields in elastic cracked domains. Figure 4.a shows an cracked elastic rectangular plate subjected to constant tractions along its vertical axis. The appropriate considerations regarding the simmetry allow the simplification of the domain (figure 4.b) by including the corresponding new boundary conditions.

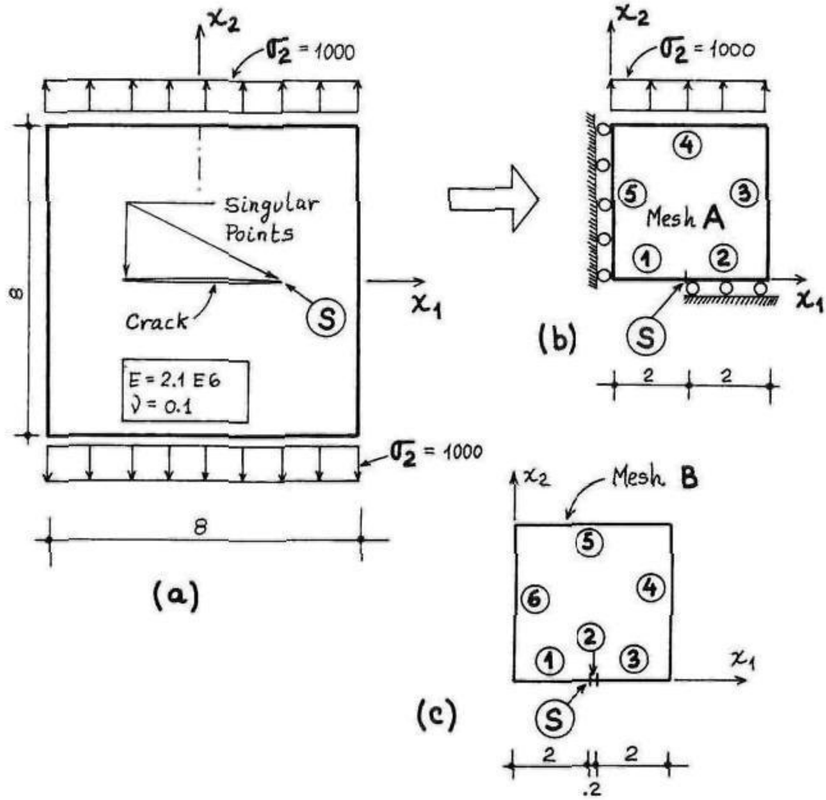


Figure 4

a) Geometry and boundary conditions

b) Simplified domain of cracked plate

c) Refined discretisation of cracked plate

This case was analysed with two base meshes: mesh

A defined with five macro-elements (figure 4.b) and mesh B defined with five macro-elements plus a small one in the vicinity of the strong singularity (point S as shown in fig. 4.c). Figure 5 compares the rates of convergence for both the p- and h-refinement.

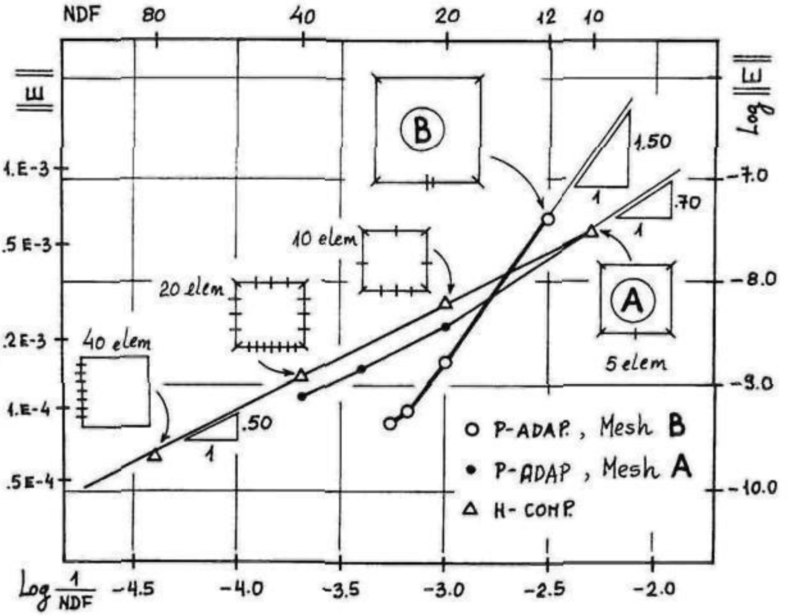


Figure 5
Comparison between global rates of convergence. Cracked plate.

The p-adaptive convergence rate over mesh B gives a value of 1.50, which is 3.0 times faster than the h-convergence one. It is not the same with regards to the p-adaptive refinement over Mesh A, practically showing no improvement when compared against the h-refinement convergence. This is due to the fact that the singularity S strongly determines the p-adaptive behaviour thus precluding a suitable polynomial fitting, especially over element number 2. However, the inclusion of a small enough element in the neighbourhood of the singular point "liberates" the development of the singularity, thus allowing the expected polynomial fitting over the element of concern (element number 3, fig. 4.c). Also observe that such a small element is only needed on the side of the singularity where the singular value is expected.

On the other hand, it seems interesting to evaluate the stress concentration factor (KI) in the first mode crack. In the available technical literature (see, e.g., Broek |7|) we can find the formulae needed to compute the stress values in the neighbourhood of the crack tip. Thus, the stress σ_2 has the following expression (ignoring higher order terms)

$$\sigma_2 = \frac{K_I}{\sqrt{2\pi \cdot r}} \cos(\theta/2) \left[1 + \mathrm{sen}\,\frac{\theta}{2} \cdot \mathrm{sen}\,\frac{3\theta}{2} \right] \qquad (5)$$

where $\tilde{r}$ is the position vector from the crack tip to the point under consideration, the θ angle is measured counterclockwise from the axis x_1 to $\tilde{r}$ vector and KI - is the stress concentration factor whose value depends upon both the geometry and boundary conditions of the panel (see Rooke et. al. |23|). Once the numerical values of the stresses σ_2 have been computed, it is possible to numerically determine the KI factor by setting $\theta=0$ in expression (5) which becomes

$$K_I = \sigma_2 \cdot \sqrt{2\pi r} \qquad (6)$$

Figure 6 shows the evolution of expression (6) as a function of the distance from the crack tip (r) in - its neighbourhood. The KI value is then obtained by extrapolating the parabola which best interpolates the - aforementioned values until its intersection with the line r=0.

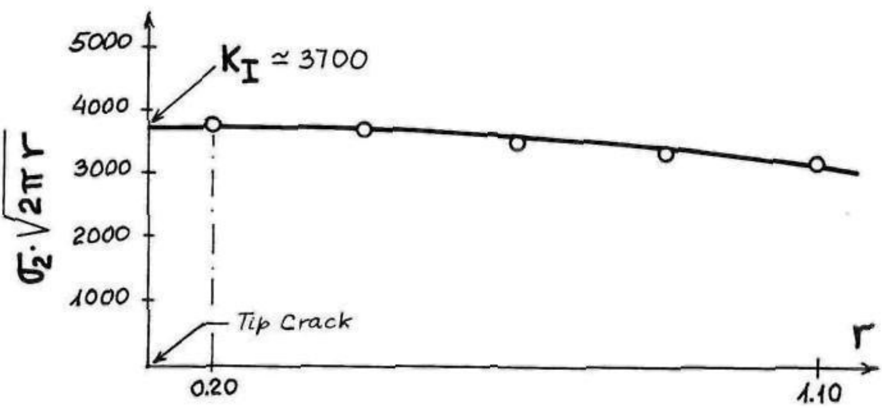


Figure 6
Evaluation of stress concentration factor. Cracked plate.

The KI obtained from figure above is approximately 3700 which compares well enough with the theoretical value of 3510, yielding a relative error of 5%.

CONCLUSIONS

The flexibility and potency of the p-adaptive BEM version proposed herein have been shown through the - analysis and discussion of the previous case studies.

It is possible to analyze non-trivial problems with a minimum of both human and machine effort, even in the presence of strong singularities.

The computational codes developed especially for this research are totally "user-friendly" and they - allow the analyst to perform the desired pursuit of - the refinement process.

The mathematical criteria developed to control - the p-adaptive version has shown a highly satisfactory performance.

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